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TEMPERATURE CONTROL OF POISEUILLE FLOW WITH PARTICULAR APPLICATION TO FLOW OF MOLTEN GLASS

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Abstract—The problem considered is that of controlling the temperature distribution of a fluid in fully developed laminar flow through a circular tube. It is assumed that suitable heating and cooling arrangements are available in a refractory casing surrounding the tube and that in its passage through a section of the tube of given length the mean temperature of the fluid has to be reduced (or increased) by a specified amount. The main feature of the problem lies in the additional requirement that the temperature distribution at the final cross-section must also be controlled to achieve some desired distribution. The problem is solved by application of optimal control theory. A quadratic cost functional is set up which includes a term measuring the closeness of fit to the desired exit temperature distribution and a constraint term to avoid unrealistic temperatures in the refractory. The solution is obtained by use of a Matrix–Riccati algorithm and also by dynamic programming. Numerical results are presented for a particular flow of molten glass. Typical exit and refractory temperature distributions are exhibited and the benefits of control are demonstrated. The results show clearly the extent to which the desired temperature distribution can be achieved for given available power.

NOMENCLATURE

$\mathbf{A}$,	system matrix;
$\mathbf{B}$,	driving matrix;
a,	inner radius of refractory;
b,	outer radius of refractory;
$\mathbf{C}$,	discrete system matrix, equation (32);
$\mathbf{D}$,	discrete driving matrix, equation (32);
$\mathbf{F}$, G,	defined in equations (34) and (35);
h,	heat-transfer coefficient;
I,	index, defined in equation (36);
k,	glass thermal conductivity;
k_r,	refractory thermal conductivity;
l,	tube length;
$\mathbf{M}$,	defined in equation (35);
$\mathbf{P}$,	solution of Matrix–Riccati equation [section 3.1 or equation (35)];
R,	control weighting parameter, equation (30);
r,	radial position;
$\mathbf{S}$,	target set weighting matrix;
s_{ii},	diagonal elements of $\mathbf{S}$;
T,	temperature;
T_j,	temperature at centre line of jth shell;
T_a, T_b,	temperature at inner and outer surfaces of refractory;
T_∞,	ambient temperature;
$\bar{T}$,	mixed mean temperature;
t,	normalized distance along tube;
u,	control variable, T_b;
W,	centre-line velocity;
$w(r,z)$,	velocity;
$\mathbf{x}$,	state vector, $[T_1, \ldots, T_n]^T$;
$\mathbf{y}$,	$\mathbf{x}-\mathbf{x}^R$;
Z,	defined in equations (34) and (35);
z,	distance along tube.

Greek symbols

$\boldsymbol{\alpha}^T$,	weighting constants for $\bar{T}$ evaluation;
α, β,	constants defined in equation (14);
β_j,	$1-(r_j^2/a^2)$;
γ, δ,	constants defined in equations (23) and (24);
ξ,	defined in Section 3.1;
η_j,	$\bar{T}-x_j^R$.

Superscripts

C,	controlled;
R,	reference or uncontrolled.

Subscripts

i,	inlet;
f,	final.

1. INTRODUCTION

THERE are many industrial processes in which it is required to heat or cool a fluid as it flows through a tube. Often, it is the increase or decrease of the mean cross-sectional temperature which is of interest, but there are cases in which it is also necessary to exercise some control over the temperature distribution, particularly that at the tube exit. Such a situation arises in the glass industry, for example, where glass nearing the final stages of certain manufacturing processes must be brought to a temperature as near uniform as possible to prevent unwanted inhomogeneities in the finished article.

This paper is concerned with setting up and analysing a simple model of a problem of this kind. Essentially, we shall consider a fluid flowing through a circular tube in which it is required to reduce the overall temperature by a specified amount and in such a way that the final temperature profile is as near uniform as possible. It is, perhaps, fairly clear intuitively that with no constraints and with the expenditure of sufficient effort, in the form of power input and output at appropriate positions along the wall of the tube, it should be possible to exercise control of the exit temperature profile to any specified degree. However, in reality there are obviously physical constraints on the temperatures which the system can withstand and limitations on the available power. Our main objective is to determine the optimum exit profile that can be achieved under certain operating conditions and the corresponding power consumed. Using a suitable measure of the nearness of the final profile to that desired, we can then investigate the variation of the return that can be achieved as the power available is varied.

The problem that we pose falls within the realm of the theory of control of distributed parameter systems. This is a very rapidly expanding area of control theory in which the study of heat flow problems has featured strongly since Butkovsky's early work on temperature distribution in a moving flat plate (see Butkovsky [1], for example). Much of the published work is concerned with the simple one-dimensional heat flow equation and, so far as we are aware, the coupled problem of the control of the temperature distribution in a fluid with spatial variations of velocity and temperature dependent viscosity has not been solved. A fairly recent survey of the literature is given by Robinson [2].

The mathematical analysis is based upon a simple model of Poiseuille flow and is aimed at bringing the problem into a form suitable for application of two standard techniques in control theory, namely the Matrix–Riccati formulation and the method of dynamic programming. In practice, dynamic programming proves to be the superior method as it is capable of handling a wider range of operating conditions. However, both methods will be discussed since it is useful to understand the limitations of the Matrix–Riccati method in the context of this problem and it also provides a valuable check on the accuracy of the results obtained by dynamic programming in the range where both are applicable.

2. MATHEMATICAL MODEL

2.1. *Flow and heat transfer equations*

Consider an incompressible fluid in steady laminar flow along a horizontal circular tube of radius a. The wall of the tube is assumed to be of negligible thickness and the tube is surrounded by a cylindrical shell of uniform insulating material of inner radius a and outer radius b. The configuration is shown in Fig. 1(a); r and z are the radial and axial coordinates, respectively, with z increasing in the direction of the flow.

Let the fluid entering the cross-section at $z = 0$ have a uniform temperature T_i. Heat is lost through the insulator and hence at all cross-sections $z > 0$ the average fluid temperature will be less than T_i. The

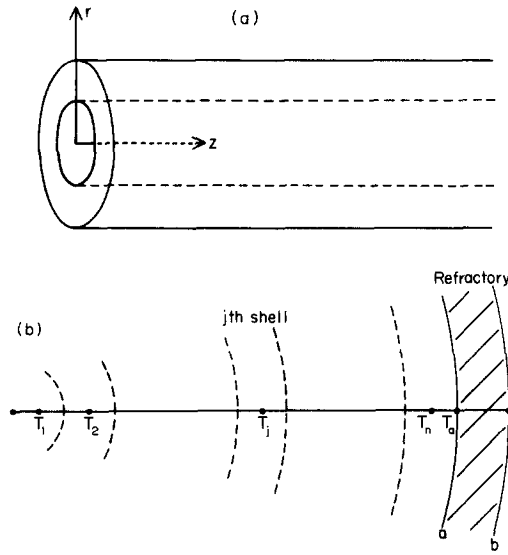

FIG. 1. (a) Cylindrical polar coordinates. (b) Illustration of radial steps.

"average temperature" may be defined in several ways, but it is convenient to use the mixed mean temperature

$$\bar{T}(z) = \frac{\int_0^a wTr\,dr}{\int_0^a wr\,dr}, \tag{1}$$

where w is the axial component of fluid velocity. Clearly $\bar{T}(0) = T_i$. At the exit plane $z = z_f$, let

$$\bar{T}(z_f) = T_f.$$

For given flow and heat-transfer conditions the cross-section $z = z_f$ at which $T_i - T_f$ reaches some given value can be found. We wish to determine conditions under which, over the length z_f, a specified temperature drop $T_i - T_f$ can be achieved, with the additional requirement that the cross-sectional temperature profile at $z = z_f$ should be as near uniform as possible. To complete the formulation of the problem it is necessary to specify:

(i) flow conditions;
(ii) heat-transfer conditions;
(iii) some constraint on allowable temperatures (for practical purposes).

For the flow, the case which is of greatest practical interest for a fluid of constant properties is the Poiseuille distribution

$$w(r, z) = W\left(1 - \frac{r^2}{a^2}\right), \tag{2}$$

where W is a constant. This profile will be used throughout the present paper.

Let $T_a(z)$ and $T_b(z)$ be the temperatures at $r = a$ and $r = b$, and let the ambient temperature at large distances from the tube be T_∞. Neglecting conduction in the axial direction and denoting by k_r the conductivity of the refractory which occupies the region $a \leqslant r \leqslant b$, we find from an elementary solution of the heat-conduction equation that the rate of flow of heat through the refractory, per unit length of tube, is

$$2\pi k_r(T_a - T_b)/\log(b/a).$$

If the rate of flow of heat to the surroundings is $h(T_b - T_\infty)$, per unit length of tube, where h is a heat-transfer coefficient, then for heat balance in the steady state

$$h(T_b - T_\infty) = 2\pi k_r(T_a - T_b)/\log(b/a). \quad (3)$$

Denoting the thermal conductivity of the fluid by k, the rate of heat transfer from the fluid to the refractory, per unit length of tube is

$$-2\pi ak(\partial T/\partial r)_{r=a},$$

where T is the fluid temperature. Hence we also have

$$h(T_b - T_\infty) = -2\pi ak(\partial T/\partial r)_{r=a}. \quad (4)$$

In the fluid, the temperature is assumed to satisfy the differential equation

$$w\frac{\partial T}{\partial z} = \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \quad (5)$$

where κ is the thermal diffusivity. This differential equation, together with conditions (3) and (4) and the inlet condition $T = T_i$, is sufficient to determine the temperature distribution at any cross-section. There is no simple analytical solution and the problem is most conveniently solved numerically. We note that $T_a(z)$ and $T_b(z)$ will be calculated as part of the solution.

Suppose that the plane $z = z_f$ at which a specified mixed mean temperature $\overline{T}_f$ is achieved has been found, assuming a reference value for the ambient temperature T_∞. Let the corresponding temperature at $r = b$ be $T_b^{(R)}$. We suppose that we have at our disposal means of controlling the temperature T_b along the wall of the refractory, and the problem is to determine the distribution $T_b^{(C)}$ of $T_b(z)$ such that, with the same inlet condition as before ($T = T_i$ at $z = 0$), the same mixed mean temperature T_f is achieved at $z = z_f$ but the temperature distribution across the plane $z = z_f$ is as near uniform as possible. It is convenient to leave until later a statement of the constraints which are imposed on $T_b^{(C)}$ to avoid unrealistic temperatures.

2.2. *Discretization scheme*

The basic differential equation (5) can be integrated by various numerical methods, but for the control problem the following finite difference scheme is convenient.

Divide the tube of radius a into n coaxial shells each of thickness $\Delta a = a/n$, the jth shell being defined by

$$(j-1)a/n \leqslant r \leqslant ja/n \quad (j = 1, 2, \ldots, n).$$

Let T_j be the temperature at the mid-point

$$r = r_j = (j-\tfrac{1}{2})\Delta a$$

of the jth shell [Fig. 1(b)]. For $2 \leqslant j \leqslant n-1$, the first and second derivatives of T at $r = r_j$ may be approximated by

$$\frac{\partial T}{\partial r} = \frac{T_{j+1} - T_{j-1}}{2\Delta a}$$

and

$$\frac{\partial^2 T}{\partial r^2} = \frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta a)^2}.$$

Using these in the differential equation (5) leads to

$$w(r_j, z)\frac{dT_j}{dz} = \frac{\kappa}{(\Delta a)^2}\left[\left(1+\frac{\Delta a}{2r_j}\right)T_{j+1} - 2T_j + \left(1-\frac{\Delta a}{2r_j}\right)T_{j-1}\right] \quad (2 \leqslant j \leqslant n-1) \quad (6)$$

At the axis, $\partial T/\partial r = 0$, and hence for $j = 1$ the differential equation (5) becomes

$$w(r_1, z)\frac{dT_1}{dz} = \kappa\left(\frac{\partial^2 T}{\partial r^2}\right)_{r=r_1} \approx \frac{\kappa}{\Delta a}\left[\left(\frac{\partial T}{\partial r}\right)_{r=r_2} - \left(\frac{\partial T}{\partial r}\right)_{r=r_1}\right] \approx \frac{\kappa}{(\Delta a)^2}\frac{T_3 - T_1}{2}. \quad (7)$$

When $j = n$, we may take

$$\frac{\partial T}{\partial r} = \frac{T_a - T_n}{\Delta a/2}$$

and

$$\frac{\partial^2 T}{\partial r^2} = \frac{1}{\Delta a}\left[\left(\frac{\partial T}{\partial r}\right)_{r=r_n} - \left(\frac{\partial T}{\partial r}\right)_{r=r_{n-1}}\right] = \frac{1}{\Delta a}\left[\frac{T_a - T_n}{\Delta a/2} - \frac{T_n - T_{n-2}}{2\Delta a}\right].$$

Hence (5) becomes

$$w(r_n, z)\frac{dT_n}{dz} = \frac{\kappa}{(\Delta a)^2}\left[2\left(1+\frac{\Delta a}{r_n}\right)T_a - \left(\frac{5}{2}+\frac{2\Delta a}{r_n}\right)T_n + \tfrac{1}{2}T_{n-2}\right]. \quad (8)$$

In the case considered here, the fluid velocity is taken to have the Poiseuille distribution given by (2). Hence the convection terms in equations (6), (7) and (8) can be replaced by

$$w(r_j, z)\frac{dT_j}{dz} = W\beta_j\frac{dT_j}{dz}, \quad (j = 1, 2, \ldots, n) \quad (9)$$

where

$$\beta_j = 1 - (r_j^2/a^2). \quad (10)$$

Based upon equations (3) and (4), there are simple relationships between T_n, T_a, T_b and T_∞ which have to be incorporated in order to define the control problems. These are

$$T_a = \frac{\alpha}{\alpha+\beta}T_b + \frac{\beta}{\alpha+\beta}T_n, \quad (11)$$

$$T_b = \frac{\alpha\beta}{\alpha+\beta(1+\alpha)}T_n + \frac{\alpha+\beta}{\alpha+\beta(1+\alpha)}T_\infty \quad (12)$$

and

$$T_a = \frac{\beta(1+\alpha)}{\alpha+\beta(1+\alpha)}T_n + \frac{\alpha}{\alpha+\beta(1+\alpha)}T_\infty \quad (13)$$

where

$$\alpha = \frac{2\pi k_r}{h \log(b/a)} \quad \text{and} \quad \beta = \frac{4\pi a k}{h \Delta a}. \tag{14}$$

It would be mathematically feasible to consider control by varying T_a, T_b or T_∞, but we give here only the scheme for control by T_b, the temperature at the outside edge of the refractory; this is the case we have studied in most detail and it is the most realistic one from a practical point of view.

2.3. *The state equations*

It will be found convenient to introduce a non-dimensional parameter t, defined by

$$t = \kappa z/Wa^2. \tag{15}$$

To conform with standard terminology in control theory, we shall refer to t as the time variable. If the overall length of the tube is l, t ranges from zero to

$$t_f = \kappa l/Wa^2. \tag{16}$$

Using equations (9)–(11) and (15), the state equations (6)–(8) can be expressed in the matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \tag{17}$$

where

$$\mathbf{x} = [T_1, T_2, \ldots, T_n]^T, \tag{19}$$

$$u = T_b, \tag{20}$$

$$\mathbf{A} = n^2 \begin{bmatrix} \frac{1}{2\beta_1} & 0 & \frac{1}{2\beta_1} & & & & \\ \frac{1}{\beta_2}\left(1-\frac{\Delta a}{2r_2}\right) & \frac{-2}{\beta_2} & \frac{1}{\beta_2}\left(1+\frac{\Delta a}{2r_2}\right) & \vdots & & & \\ & \cdots \frac{1}{\beta_j}\left(1-\frac{\Delta a}{2r_j}\right) & & -\frac{2}{\beta_j} & \frac{1}{\beta_j}\left(1+\frac{\Delta a}{2r_j}\right)\cdots & & \\ & & & \vdots & \frac{1}{2\beta_n} & 0 & \frac{\gamma}{\beta_n} \end{bmatrix} \tag{21}$$

and

$$\mathbf{B} = n^2[0, \ldots, 0, \delta/\beta_n]^T; \tag{22}$$

the quantities γ and δ are defined by

$$\gamma = 2\left(1+\frac{\Delta a}{r_n}\right)\frac{\beta}{\alpha+\beta} - \left(\frac{5}{2}+2\frac{\Delta a}{r_n}\right) \tag{23}$$

and

$$\delta = 2\left(1+\frac{\Delta a}{r_n}\right)\frac{\alpha}{\alpha+\beta}. \tag{24}$$

The vector $\mathbf{x}$ is called the *state vector*, u is the control variable, $\mathbf{A}$ is the *system matrix* and $\mathbf{B}$ is the *driving matrix*.

2.4. *The cost functional*

Once a suitable reference temperature distribution $u = u^R$ is defined, equation (17) may be integrated to give the reference state $\mathbf{x} = \mathbf{x}^R$. Thus u^R and $\mathbf{x}^R$ satisfy

$$\dot{\mathbf{x}}^R = \mathbf{A}\mathbf{x}^R + \mathbf{B}u^R, \tag{25}$$

with $\mathbf{x}(0) = \mathbf{x}_0$ (given). If u is now changed from u^R, the perturbed solution $\mathbf{x}$ will satisfy the equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \tag{26}$$

with $\mathbf{x}(0) = \mathbf{x}_0$. If

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^R \quad \text{and} \quad \Delta u = u - u^R, \tag{27}$$

then

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\Delta u \tag{28}$$

with $\mathbf{y}(0) = \mathbf{0}$.

Now, with the reference control u^R applied, the output mixed mean temperature $\overline{T}(t_f)$ may be expressed as

$$\overline{T}(t_f) = \boldsymbol{\alpha}^T \mathbf{x}^R(t_f),$$

where $\boldsymbol{\alpha}^T$ is an n-vector of weighting constants. The aim is to improve the temperature distribution at the exit plane by varying T_b but to maintain $\overline{T}(t_f)$ as in the reference condition. A suitable cost functional is therefore the quadratic criterion

$$J_1 = \tfrac{1}{2}\sum_{j=1}^{n} s_j(\overline{T}-x_j)^2|_{t=t_f}$$

$$= \tfrac{1}{2}\sum_{j=1}^{n} s_j(\overline{T}-x_j^R-y_j)^2|_{t=t_f},$$

where $s_1, s_2, \ldots, s_n$ are weighting constants. If we write

$$\overline{T} - x_j^R = \eta_j, \tag{29}$$

then this defines the target set in the control problem, and

$$J_1 = \tfrac{1}{2}(\boldsymbol{\eta}-\mathbf{y})^T\mathbf{S}(\boldsymbol{\eta}-\mathbf{y})|_{t=t_f},$$

where $\mathbf{S} = \operatorname{diag}(s_j)$. This gives a terminal cost functional. However, the problem is not well-posed without some form of constraint on the control. The cost functional J_1 is therefore replaced by

$$J = \tfrac{1}{2}(\boldsymbol{\eta}-\mathbf{y})^T\mathbf{S}(\boldsymbol{\eta}-\mathbf{y})|_{t=t_f} + \tfrac{1}{2}\int_0^{t_f} R(\Delta u)^2\,dt. \tag{30}$$

The weighting matrix $\mathbf{S}$ and the scalar R are free design parameters which had to be chosen and adjusted in the light of experience and computed results.

3. ALGORITHMS FOR SOLVING THE CONTROL PROBLEM

3.1. *The Matrix–Riccati algorithm*

The first method tried was the continuous Matrix–Riccati algorithm which follows directly from the application of Pontryagin's minimum principle to the problem posed by equations (28) and (30). Details of the solution may be found, for example, in Sage [3].

The matrix equation

$$\dot{\mathbf{P}} = -\mathbf{PA} - \mathbf{A}^T\mathbf{P} + \mathbf{PB}R^{-1}\mathbf{B}^T\mathbf{P}$$

and the vector equation

$$\dot{\xi} = -[\mathbf{A} - \mathbf{B}R^{-1}\mathbf{B}^T\mathbf{P}]^T\xi$$

have to be integrated backwards in time from final conditions

$$\mathbf{P}(t_f) = \mathbf{S} \quad \text{and} \quad \xi(t_f) = \mathbf{S}\eta(t_f).$$

The control function is then given by

$$\Delta u(t) = -R^{-1}(t)\mathbf{B}^T(t)[\mathbf{P}(t)\mathbf{y}(t) - \xi(t)],$$

and the state is obtained by integrating the equation

$$\dot{\mathbf{y}} = \mathbf{Ay} + \mathbf{B}\Delta u$$

forwards in time from the initial condition $\mathbf{y}(0) = \mathbf{0}$. Finally, we can obtain

$$\mathbf{x} = \mathbf{x}^R + \mathbf{y} \quad \text{and} \quad u = u^R + \Delta u.$$

The first step in the procedure was to assume a constant ambient temperature $T_\infty = 20°\text{C}$ and then to use a modified system matrix [incorporating equation (13) instead of (11) in equation (8)] to calculate the corresponding reference temperature u^R at the outer edge of the refractory and the reference temperature distribution $\mathbf{x}^R$ in the glass. Attention was then turned to the application of the Matrix–Riccati algorithm.

The weighting matrix $\mathbf{S}$ was chosen to be of diagonal form with equal diagonal elements s_{ii} so that the ratio R/s_{ii} was the only design parameter which needed to be considered. Without loss of generality we could therefore put $s_{ii} = 100$. By decreasing R greater weight was put upon the cost of errors in the exit temperature at the expense of increased control temperature for which greater heating or cooling rates were needed. In most of the calculations we took $n = 5$.

The algorithm turned out to be useful but not completely satisfactory because of difficulties with numerical stability and accurate results were obtained only for the range of R values greater than 5. The source of the difficulty was taken to be due to the very large spread of eigenvalues of the system matrix, which were calculated to be -0.852, -17.91, -58.09 and -114.64 ± 31.43. This made it necessary to use quite small time-steps for acceptable accuracy; typically, 180 time-steps were needed when $n = 5$ and about 1500 time-steps when $n = 10$. By using time-steps of variable lengths, it was possible to reduce the number to about 900 when $n = 10$, but this was judged to be still too many for this approach to be economic for such cases. However, whenever results were obtained which overlapped the dynamic programming results, agreement was excellent and gave valuable supporting information.

3.2. *Discrete dynamic programming approach*

The problem posed in equations (28) and (30) differs only slightly from standard problems by virtue of the target set $\eta(t_f)$. In spite of this, we could not find the solution of this particular problem in the literature. The closest approach appears to be that of Kleindorfer [4] who included linear terms in the cost functional. However, the precise equivalence between his algorithm and that used here is hard to establish because of notational difficulties. The main features and results are as follows.

The problem is first converted to one in discrete time by dividing the time interval $[0, t_f]$ into N equal intervals h. The differential equation (28) is replaced by the difference equation

$$\mathbf{y}_{k+1} = \mathbf{C}\mathbf{y}_k + \mathbf{D}u_k, \tag{31}$$

where

$$\mathbf{C} = \mathbf{I} + h\mathbf{A} \quad \text{and} \quad \mathbf{D} = h\mathbf{B}. \tag{32}$$

The cost functional J is replaced by the sum

$$J = \tfrac{1}{2}(\eta - \mathbf{y}_N)^T\mathbf{S}(\eta - \mathbf{y}_N) + \tfrac{1}{2}\sum_{k=0}^{N-1}(\Delta u)_k^2 R', \tag{33}$$

where

$$R' = hR.$$

If we define Q_N, Q_{N-k} and S_{N-k} in the following way:

$$Q_N = \tfrac{1}{2}(\eta - \mathbf{y}_N)^T S_N(\eta - \mathbf{y}_N),$$

$$Q_{N-k} = \tfrac{1}{2}R'U_{n-k}^2 + Q_{N-k+1},$$

and $S_{N-k} = \min Q_{N-k}$ over $\{U_{N-k}, U_{N-k+1}, \ldots, U_{N-1}\}$, then we can set up the recurrence relations

$$(\Delta u)_{N-k} = \mathbf{F}(N-k)\mathbf{y}_{N-k} + G(N-k) \tag{34}$$

and

$$S_{N-k}(\mathbf{y}_{N-k}) = \tfrac{1}{2}\mathbf{y}_{N-k}^T\mathbf{P}(k)\mathbf{y}_{N-k} + \tfrac{1}{2}\mathbf{y}_{N-k}^T\mathbf{M}(k) + \tfrac{1}{2}\mathbf{M}^T(k)\mathbf{y}_{N-k} + \tfrac{1}{2}Z(k). \tag{35}$$

where

$$\mathbf{F}(N-k) = -(R' + \mathbf{D}^T\mathbf{P}(k-1)\mathbf{D})^{-1}\mathbf{D}^T\mathbf{P}(k-1)\mathbf{C},$$

$$G(N-k) = -\tfrac{1}{2}(R' + \mathbf{D}^T\mathbf{P}(k-1)\mathbf{D})^{-1} \times [\mathbf{D}^T\mathbf{M}(k-1) + \mathbf{M}^T(k-1)\mathbf{D}],$$

$$\mathbf{P}(k) = \mathbf{F}^T(N-k)R'\mathbf{F}(N-k) + [\mathbf{C} + \mathbf{DF}(N-k)]^T\mathbf{P}(k-1) \times [\mathbf{C} + \mathbf{DF}(N-k)],$$

$$\mathbf{M}(k) = \mathbf{F}^T(N-k)R'G(N-k) + [\mathbf{C} + \mathbf{DF}(N-k)]^T \times [\mathbf{P}(k-1)\mathbf{D}G(N-k) + \mathbf{M}(k-1)],$$

and

$$Z(k) = Z(k-1) + G(N-k)RG(N-k) + G(N-k)\mathbf{D}^T\mathbf{P}(k-1)\mathbf{D}G(N-k) + G(N-k)\mathbf{D}^T\mathbf{M}(k-1) + \mathbf{M}^T(k-1)\mathbf{D}G(N-k).$$

The initial conditions are

$$\mathbf{P}(0) = \mathbf{S}, \ \mathbf{M}(0) = -\mathbf{S}\eta \ \text{ and } \ Z(0) = \eta^T\mathbf{S}\eta.$$

The values of $\mathbf{P}$, $\mathbf{M}$, $\mathbf{F}$ and G can then be evaluated in the following order

$\mathbf{F}(N-1)$, $G(N-1)$, $\mathbf{P}(1)$, $\mathbf{M}(1)$; $\mathbf{F}(N-2)$, $G(N-2)$, $\mathbf{P}(2)$, $\mathbf{M}(2)$, $\ldots$, $\mathbf{F}(0)$, $G(0)$.

Using the initial condition $\mathbf{y}_0 = \mathbf{0}$, $(\Delta u)_0$ may be evaluated from

$$(\Delta u)_0 = \mathbf{F}(0)y_0 + G(0).$$

Then $\mathbf{y}_1 = \mathbf{C}\mathbf{y}_0 + \mathbf{P}(\Delta u)_0$, and all subsequent values of $(\Delta u)_k$ and $\mathbf{y}_k$ can be determined from equations (31) and (34). The actual state and control variables are evaluated from

$$\mathbf{x} = \mathbf{x}^R + \mathbf{y} \quad \text{and} \quad u = u^R + \Delta u.$$

Practical experience with this algorithm showed it to be more robust than the Matrix–Riccati approach. The remarkably small number of ten time steps was sufficient to give acceptably accurate outlet temperature profiles. The control temperatures however had a spurious oscillation superimposed, the amplitude of which decreased rapidly upon increasing the number of time-steps. Results were regarded as satisfactory when the number of time-steps was increased to 80; this number of time-steps was taken for all subsequent cases with $n = 5$. When $n = 10$, sufficient accuracy was achieved with 160 time-steps.

4. RESULTS

4.1. *Data*

The methods of the foregoing sections were applied to a case typical to the glass industry in which glass is fed along a circular tube to a machine for pressing lens blanks. For good optical qualities it is desirable that the temperature of the glass emerging from the tube should be as near uniform as possible. The data which follows it taken from Leman [5].

Glass at a temperature of 1100°C was assumed to enter a circular tube of radius 26.7 mm and of length 813 mm surrounded by a refractory casing of outer radius 39.4 mm. The fluid velocity at the mid-point of the entrance to the tube was taken to be 19.25 mm/s. The ambient temperature used in calculating the reference trajectories was 20°C and the heat-transfer coefficient (h) was assumed to be 13.8 W/m °C. For the refractory, the thermal conductivity was taken to be 2.09 Wm/m^2 °C, and for the glass the following physical properties were assumed:

density = 2.34 km/m^3
thermal conductivity = 20.9 Wm/m^2 °C
thermal diffusivity = 6.25 mm^2/s.

With this data, the Péclet number is 82.2 and the tube aspect ratio (length/radius) is 30.5. The non-dimensional length parameter t_f may be expressed as

$$t_f = \frac{\text{aspect ratio}}{\text{Péclet number}}$$

and has the value 0.372 in this particular case.

Since, as stated in Section 3, it was possible to obtain a more complete set of results using the dynamic programming algorithm the results presented are mostly from that source. Results from the Matrix–Riccati algorithm are presented in Section 4.3 only for comparison purposes. Unless otherwise stated, $n = 5$.

4.2. *Comparison of results for different values of* R

The parameter R governs the weighting to be applied to the control temperature in the cost functional J [equation (30)]; smaller values of R allow greater variations of the control temperature and this leads to better control of the outlet temperature profile. The glass temperature profiles at the tube exit for the uncontrolled case and several controlled cases are shown in Fig. 2. For the uncontrolled case the difference between centre line temperatures and those at the outer sections of the tube is about 55°C. Even a modest amount of control ($R = 5$) improves upon this, reducing the temperature difference to around 17°C. By decreasing R further the temperature difference can be made as small as desirable; for example, with $R = 0.1$ the difference is only 4°C.

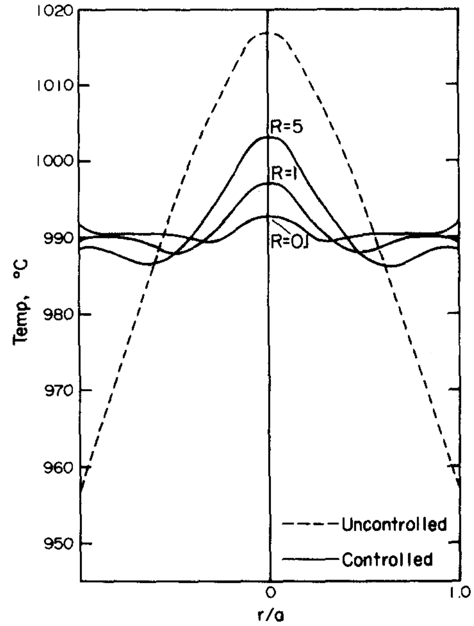

FIG. 2. Glass temperature profiles at the exit plane.

The corresponding control temperatures to achieve these glass temperature profiles are shown in Figs. 3(a)–(c). In each figure the broken line represents the temperature T_b for the uncontrolled case. All the controlled cases in comparison show the same pattern with extra cooling over the first three-quarters of the tube length and extra heating over the last quarter. The physical interpretation is that it is necessary to cool the central core of the glass initially to achieve a temperature closer to the mixed mean temperature and then to put heat back into the outer layers in the final stages as the glass approaches the exit plane. The slight down-turn observed in Figs. 3(a)–(b) are needed to prevent the outermost sections being overheated. As R is decreased to small values the system operates so as to try and eliminate all the minor variations in the glass temperature profile. The control temperature graph in Fig. 3(c) shows an additional change of direction as a result.

The actual peak temperatures required are regarded as being feasible in practice.

4.3. *The error due to radial discretization*

The majority of the results are given for the case when five radial steps were taken ($n = 5$). For comparison, check calculations with ten radial steps give the results shown in Figs. 4(a) and (b) for the case

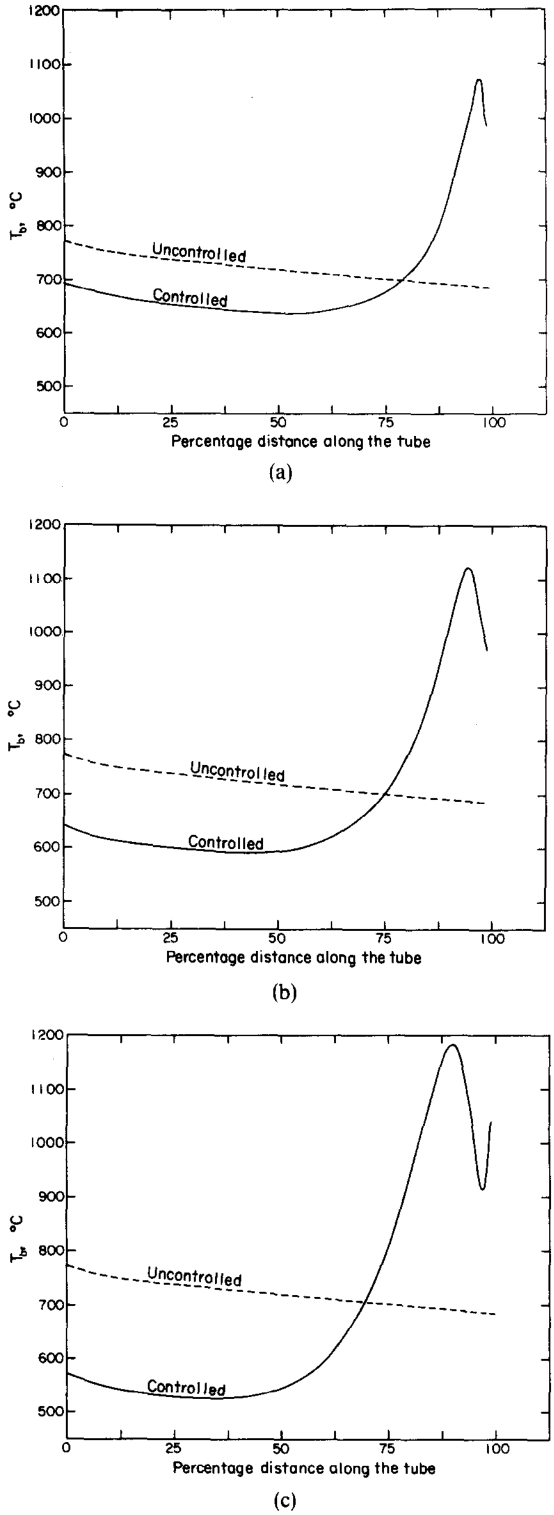

FIG. 3. Refractory temperatures for control, T_b. (a) $R = 5.0$. (b) $R = 1.0$. (c) $R = 0.1$.

$R = 5$. The larger number of radial steps is seen to give the flatter glass temperature profile, the maximum discrepancy being about 4°C. Whilst this is appreciable from a numerical computation viewpoint it is not practically significant and since the smaller number of steps provides a useful economy in computation time it was used as the basis for all other computations.

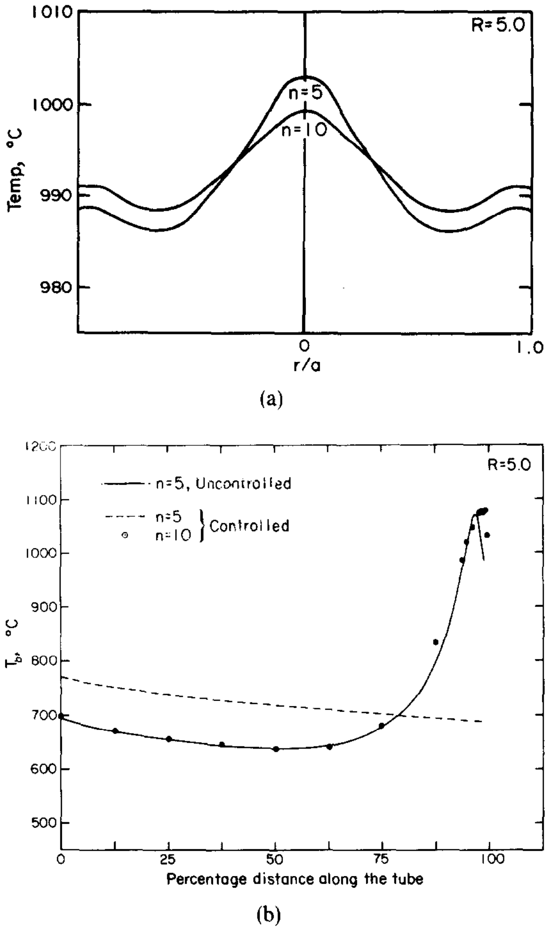

FIG. 4. The effect of increasing the number of radial steps. (a) Glass temperature profiles at the exit plane. (b) Refractory temperatures for control, T_b.

Further, the calculations with $n = 5$ provide a conservative estimate of what can be achieved by control.

The control temperatures (Fig. 4b) for the two values of n differ only in the details over the last 10 per cent of the tube length.

4.4. *Comparison of results from Matrix–Riccati and dynamic programming algorithms*

The calculations for $n = 5$ using the Matrix–Riccati algorithm completely support those obtained by dynamic programming. The glass temperature profiles differ by less than 1.5°C (Fig. 5a). The control profiles (Fig. 5b) are also nearly identical with slight differences occurring only over the last 10 per cent of the tube length.

4.5. *The effect of tube length*

For all the results quoted so far the normalized length of tube was held constant. To investigate how much design flexibility was available in tube length and the other parameters involved, calculations for the case $R = 1.0$ were repeated for normalized tube lengths of three quarters and one and a quarter times the original tube length. In each case the aim was to achieve the same mixed mean temperature.

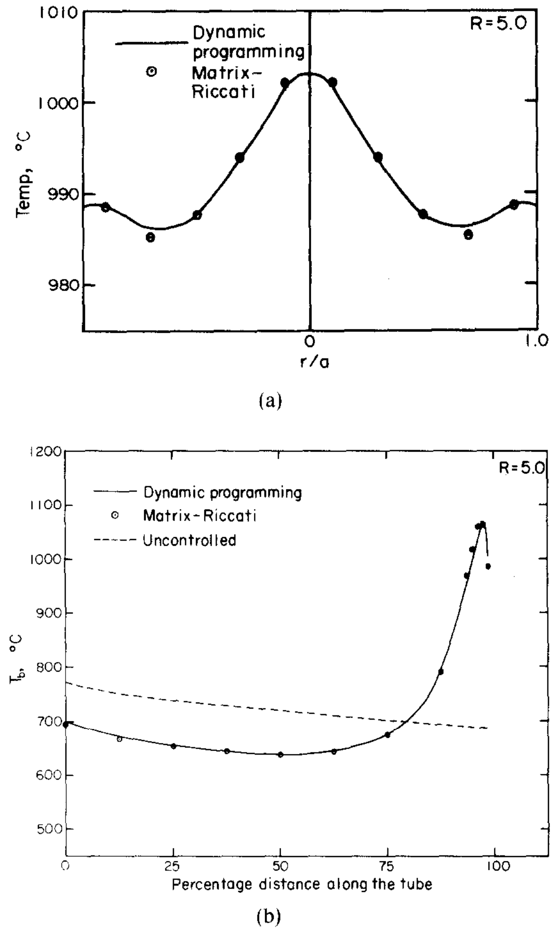

FIG. 5. Comparison between dynamic programming and Matrix–Riccati solutions. (a) Glass temperature profiles at the exit plant. (b) Refractory temperatures for control, T_b.

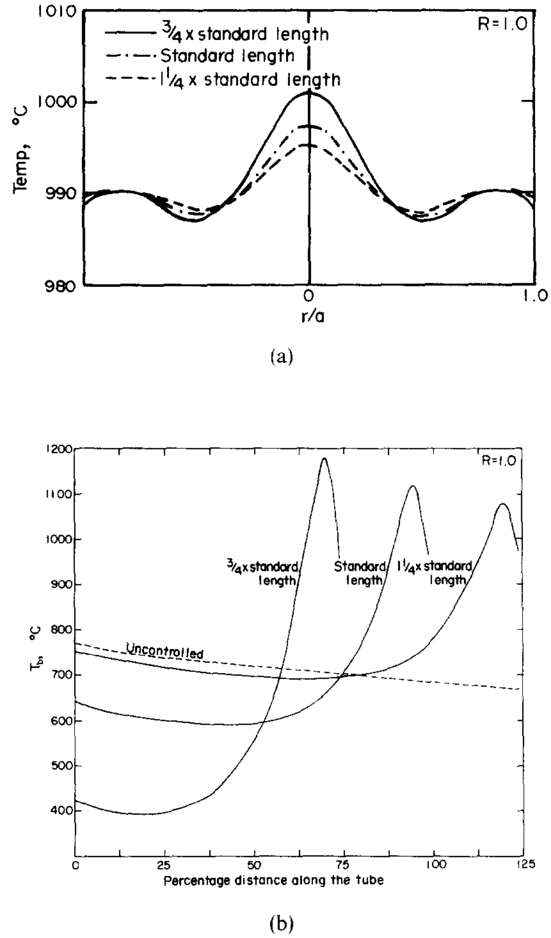

FIG. 6. The effect of varying tube length. (a) Glass temperature profiles at the exit plane. (b) Refractory temperatures for control, T_b.

With the longer tube the glass exit temperature profile was very slightly improved compared with the standard problem, the difference being about 2°C at the centre line (Fig. 6a). The control temperature profile, shown in Fig. 6(b), is similar in shape to the standard profile but is elongated. The control temperatures are also closer to the reference temperatures. With the shorter tube the exit temperature profile was rather worse than the standard case (about 5°C difference at the centre line). The control temperature profile is significantly compressed and more extreme control temperatures are demanded. In particular, along the first part of the tube the control temperatures are lower than those in the standard case by up to 350°C.

It should be borne in mind however that the final mixed mean temperature $\bar{T}$ has been maintained constant in this comparison. In the three-quarter-length solution, extra cooling is needed to bring down the average temperature to a value below that which would occur naturally at the exit of the shorter pipe without control.

4.6. *Alternative glass exit temperature profiles*

As an extension of the work, the target set was altered. In many processes it is desirable at exit for the temperature at the walls to be perhaps 50°C higher than at the centre line. To test the ability of the program to achieve this sort of profile the target set was adjusted. In the new problem the approximately parabolic profile of the uncontrolled case had to be converted to a similarly shaped profile in the controlled case with the outside edge temperature higher than the centre-line temperature.

The results shown in Fig. 7(a) show that this was achieved fairly successfully though at the expense of considerably higher control temperature variations as seen in Fig. 7(b). These could be made less extreme of course by lengthening the tube as discussed in Section 4.4. These results clearly demonstrate the feasibility of achieving a variety of temperature profiles to an acceptable degree of accuracy.

An interesting observation is that the control temperatures for this case (measured relative to the uncontrolled case) are precisely twice those needed for the original problem when we were aiming for the flat profile. The reason for this becomes apparent upon inspection of equation (30). The new target set corresponds to a doubling of the original η. If y and Δu are simultaneously doubled then they clearly provide the optimum for the revised problem.

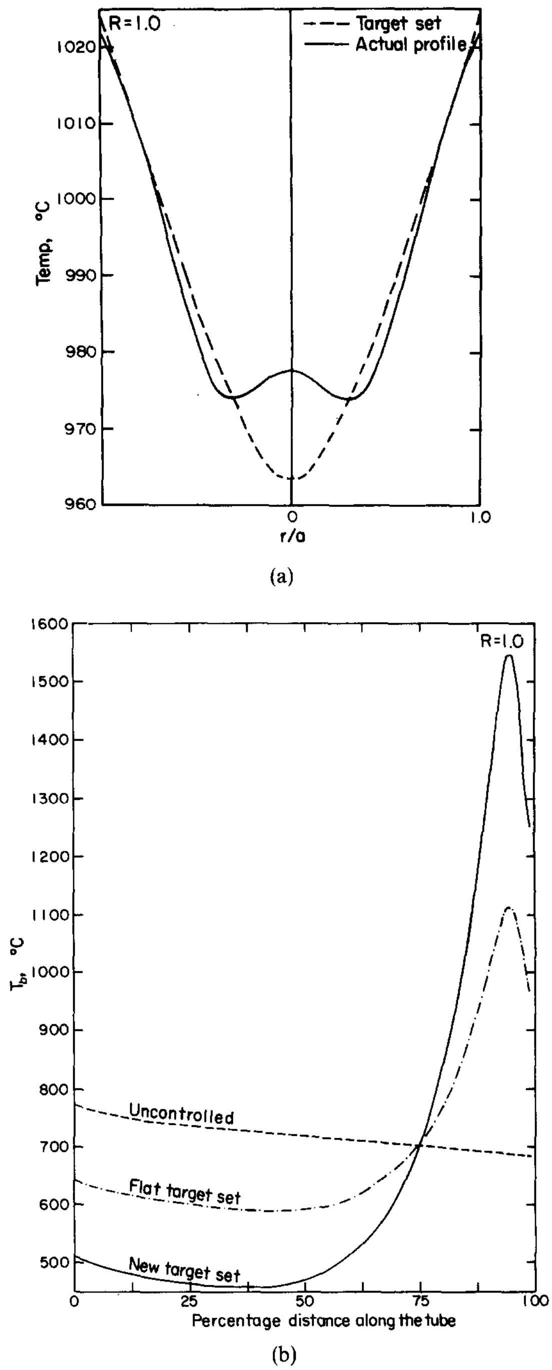

FIG. 7. Results for the revised target set. (a) Glass temperature profile at the exit plane. (b) Refractory temperatures for control, T_b.

This argument can be extended. The solution for a whole class of target sets of the same shape as the original uncontrolled profile but containing a multiplicative factor can by a suitable scaling immediately be inferred from the results for just one value of the multiplicative factor.

4.7. *Heating rates*

In the results presented so far, the control activity has been defined in terms of the temperature T_b at the outer edge of the refractory. However, a parameter which is of greater practical importance is the heating rate, since control is often exercised by heating elements embedded in the refractory casing. The heating rate corresponding to a given temperature distribution T_b is readily obtained by calculating the additional amount of heat which must be generated (or extracted) to raise (or lower) the temperature at the outer edge of the refractory from the value it would have had in the absence of control.

To give an indication of the return that can be achieved for a given expenditure of power, the power consumed may be correlated with

$$I = \left[\int_0^a r(T-\bar{T})^2\,dr\right]_C^{\frac{1}{2}} \Bigg/ \left[\int_0^a r(T-\bar{T})^2\,dr\right]_R^{\frac{1}{2}} \qquad (36)$$

where the subscripts C and R refer to the controlled and uncontrolled values, respectively. The index I is a measure of the extent to which the temperature profile at the exit departs from the required profile; the value $I = 1$ corresponds to no control, whilst $I = 0$ represents the ideal condition of perfect control.

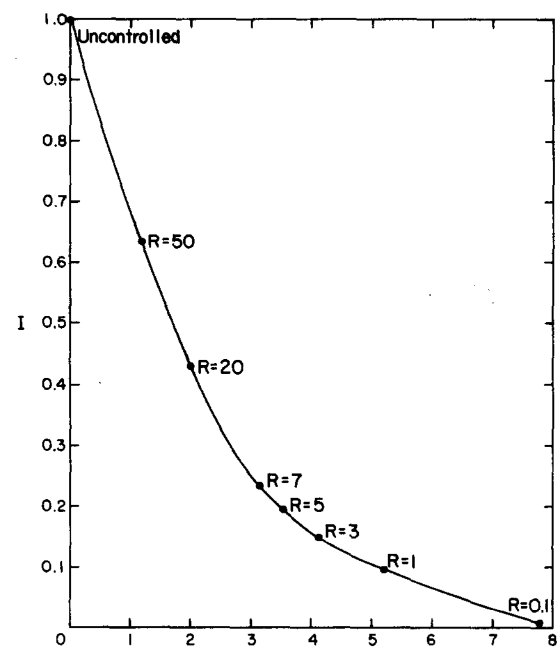

FIG. 8. Variations of index of performance, I, with power consumed.

Figure 8 shows the results corresponding to the temperature profiles of Figs. 3(a)–(c). This clearly indicated that for this problem useful control involves about 3 kW of power, and if this power is doubled to 6 kW an almost completely flat temperature profile can be achieved. An interesting incidental observation is that although the Matrix–Riccati method was successful only in the range $R \geqslant 5$, this represents a good part of the region where control of the outlet temperature profile is significant.

5. CONCLUDING COMMENTS

This problem was considered as part of a more extensive study of the dynamic control of glass flowing down a pipe. A complicating feature, not dealt with in the present paper, is that in the case of molten glass viscosity variations are likely to be significant. A scheme for dealing with this aspect of the problem has been developed. The next stage will be to consider dynamic control to take account of fluctuations in operating conditions, external disturbances, etc. Problem of observability and state reconstruction will feature at this stage.

Since the dimensionality of the problem will increase significantly it will become imperative to devise or make use of methods which economise on computer storage. From this study, the dynamic programming algorithm has emerged as the more promising of the two approaches considered and work is in hand to refine it for the next stage. The Matrix–Riccati equation was solved by the direct integration of the differential equations. Many other methods exist in the literature with improved characteristics and a study of these could be fruitful.

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CONTROLE DE LA TEMPERATURE D'UN ECOULEMENT DE POISEUILLE AVEC APPLICATION PARTICULIERE A UN ECOULEMENT DE VERRE FONDU

Résumé—Le problème considéré est celui du contrôle de la distribution de température dans un fluide en écoulement laminaire établi dans un tube circulaire. On suppose qu'un dispositif approprié de chauffage et de refroidissement est placé dans le manchon réfractaire enveloppant le tube et que la température moyenne du fluide doit être réduite (ou augmentée) d'une quantité déterminée durant son passage dans une section de longueur donnée. L'aspect essentiel du problème réside dans la condition supplémentaire portant sur la distribution de température dans la section finale qui doit être contrôlée afin d'obtenir la distribution désirée.

Le problème est résolu par application de la théorie du contrôle optimal. Une fonctionnelle de coût quadratique est construite qui comprend un terme mesurant l'écart de l'approximation obtenue par rapport à la distribution de température finale désirée et un terme de contrainte permettant d'éviter des températures du réfractaire irréalisables. La solution est obtenue par utilisation d'un algorithme matriciel de Riccati ainsi que par programmation dynamique.

Des résultats numériques sont présentés dans le cas particulier d'un écoulement de verre fondu. On a tracé les distributions types de température à la sortie et dans le réfractaire et on montre les avantages du contrôle de température. Les résultats font apparaitre clairement avec quelle approximation la distribution de température désirée peut être atteinte, pour une puissance disponible donnée.

DIE REGELUNG DER TEMPERATUR EINER POISEUILLE-STRÖMUNG IN BESONDEREM HINBLICK AUF DIE STRÖMUNG VON GESCHMOLZENEM GLAS

Zusammenfassung—Es wird das Problem der Regelung der Temperaturverteilung eines Fluides bei voll ausgebildeter laminarer Rohrströmung untersucht. Es wird angenommen, daß geeignete Heiz- und Kühlvorrichtungen in einem hitzebeständigen Gehäuse um das Rohr vorhanden sind und daß beim Durchströmen einer bestimmten Rohrstrecke die mittlere Temperatur des Fluides um einen bestimmten Betrag gesenkt (oder erhöht) werden soll. Das Hauptmerkmal des Problems liegt in der zusätzlichen Forderung, daß zur Erreichung der gewünschten Temperaturverteilung die Verteilung im Austrittsquerschnitt ebenfalls geregelt werden muß.

Das Problem wird mit Hilfe der Optimal-Regeltheorie gelöst. Es wird eine quadratische Funktion aufgestellt, die einen Term enthält, der die Anpassung an die gewünschte Austrittstemperatur-Verteilung erfaßt und einen Begrenzungsterm, der unrealistische Temperaturen im Heizelement ausscheidet. Die Lösung wird durch den Matrix–Riccati-Algorithmus und durch dynamisches Programmieren erhalten.

Es werden numerische Ergebnisse für eine spezielle Strömung von geschmolzenem Glas angegeben. Die typischen Temperaturen im Austrittsquerschnitt und in der Heizstrecke werden dargestellt und die Vorteile der Regelung aufgezeigt. Die Ergebnisse zeigen deutlich, bis zu welchem Ausmaß die gewünschte Temperaturverteilung bei gegebener Leistung erreicht werden kann.

ТЕМПЕРАТУРНЫЙ КОНТРОЛЬ ТЕЧЕНИЯ ПУАЗЕЙЛЯ ДЛЯ ЧАСТНОГО СЛУЧАЯ ПОТОКА РАСПЛАВЛЕННОГО СТЕКЛА

Аннотация — Рассматривается задача о регулировании распределения температуры жидкости при полностью развитом ламинарном течении в кольцевой трубе. Предполагается, что в жаропрочном чехле, окружающем трубу, имеются соответствующие устройства для нагревания и охлаждения и что при прохождении через участок трубы заданной длины средняя температура жидкости должна снижаться (или повышаться) на определенную величину. Основная особенность задачи заключается в дополнительном требовании о регулировании распределения температуры в конечном сечении для достижения желаемого распределения. Задача решается с помощью теории оптимального регулирования. Получен квадратичный целевой функционал, включающий член, который определяет близость пробного и заданного распределения температуры на входе, а также член, позволяющий исключить нереальные температуры в огнеупоре. Решение получено с помощью алгоритма Матрикса-Риккати и динамического программирования. Приводятся результаты для частного случая расплавленного стекла. Показаны типичные распределения температур на входе и в огнеупоре и продемонстрированы преимущества регулирования. Результаты четко показывают степень достижения необходимого температурного распределения при заданной мощности.